

Large graphs and symmetric sums of squares

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An example

Theorem (Mantel, 1907)

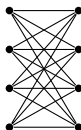
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The maximum is attained on $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$:

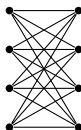


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$\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$ edges out of $\binom{n}{2}$ potential edges, no triangles.


Other triangle densities

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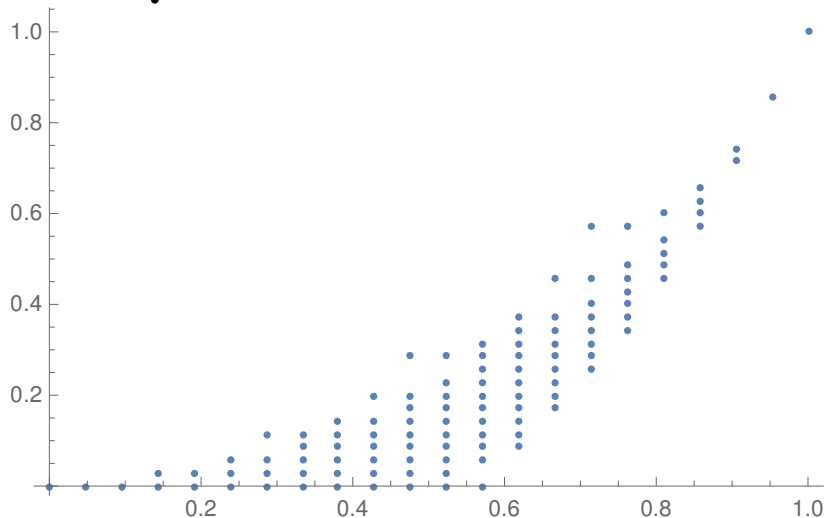
then $(d(\square, G), d(\triangle, G)) = \left(\frac{9}{\binom{7}{2}}, \frac{2}{\binom{7}{3}}\right) \approx (0.43, 0.06)$.

Is that the max edge density among graphs on 7 vertices with 2 triangles?

What can $(d(\square, G), d(\triangle, G))$ be if G is any graph on 7 vertices?

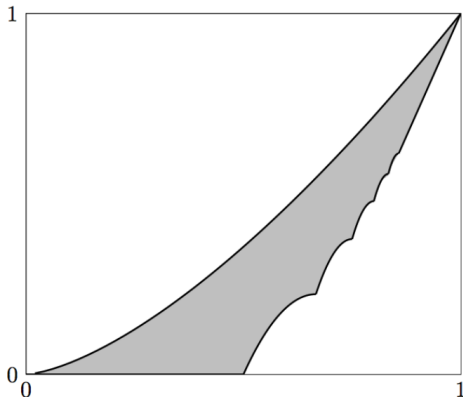
All density vectors for graphs on 7 vertices

$(d(\text{path}_2, G), d(\text{triangle}, G))$ for any graph G on 7 vertices



All density vectors for graphs on n vertices as $n \rightarrow \infty$

$(d(\text{---}, G), d(\text{---}, G))$ for any graph G on n vertices as $n \rightarrow \infty$



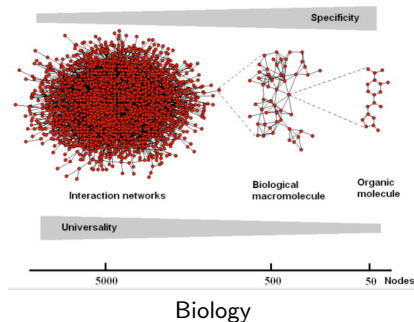
(Razborov, 2008)

Why care?

Large graphs are everywhere!

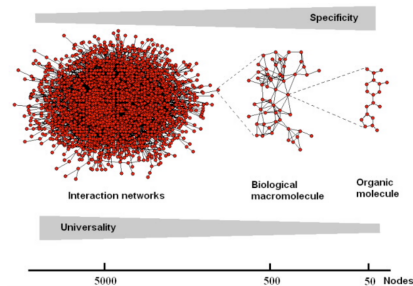
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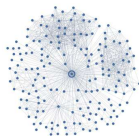


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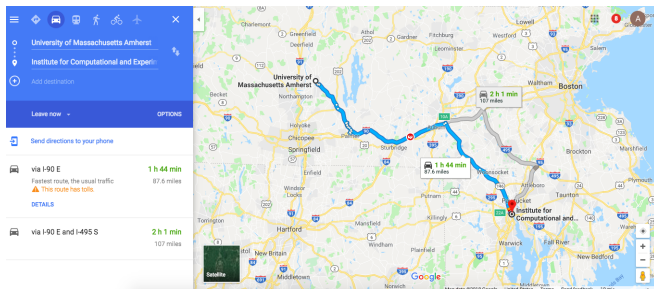


Biology



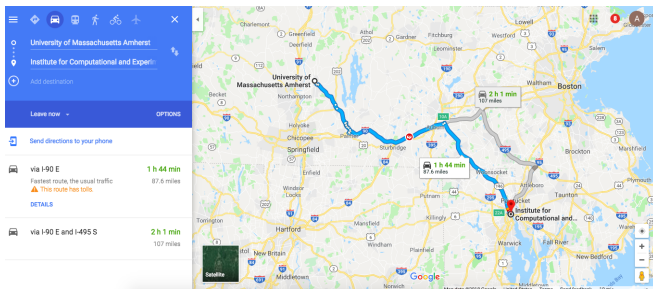
Facebook graph

More reasons to care!

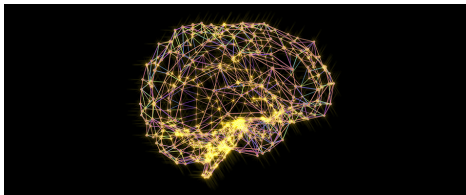


Google Maps

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Google Maps



Alfred Pasiaka/Science Photo Library/Getty Images

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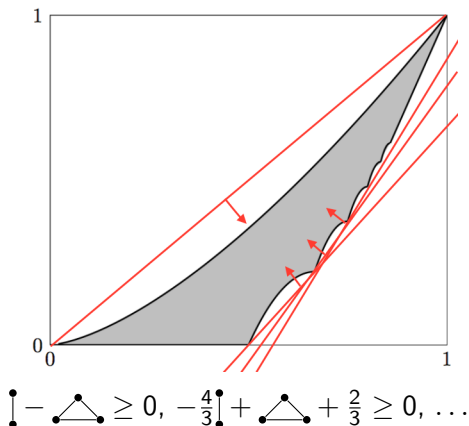
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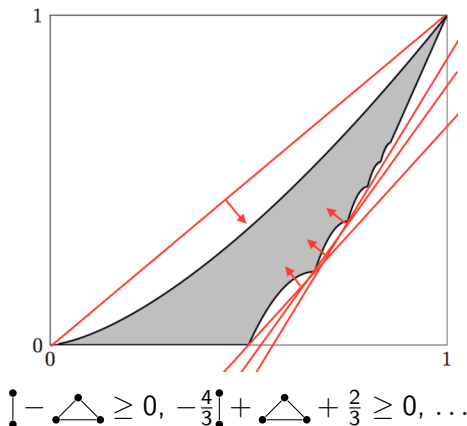
- 1 How do global and local properties relate?
- 2 What is even possible locally?

Graph density inequalities

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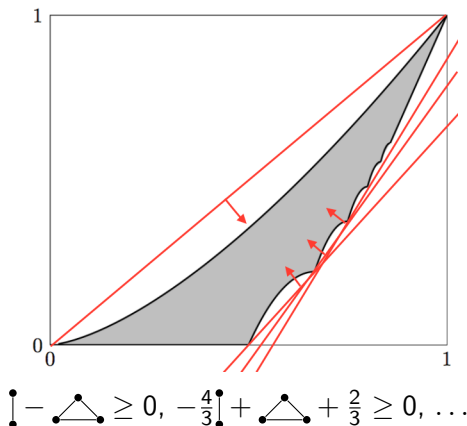


Graph density inequalities



Nonnegative polynomial graph inequality: a polynomial* involving **any** graph densities (not just edges and triangles, and not necessarily just two of them) that, when evaluated on any graph on n vertices where $n \rightarrow \infty$, is nonnegative.

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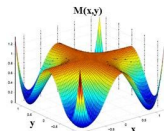
Nonnegative polynomial graph inequality: a polynomial* involving **any** graph densities (not just edges and triangles, and not necessarily just two of them) that, when evaluated on any graph on n vertices where $n \rightarrow \infty$, is nonnegative. **How can one certify such an inequality?**

Certifying polynomial inequalities



Certifying polynomial inequalities

A polynomial $p \in \mathbb{R}[x_1, \dots, x_n] =: \mathbb{R}[\mathbf{x}]$
is **nonnegative** if $p(x_1, \dots, x_n) \geq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

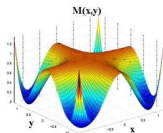


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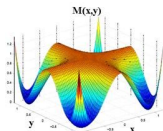


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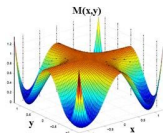
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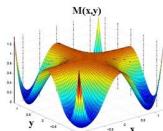
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Motzkin (1967, with Taussky-Todd): $M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is a nonnegative polynomial but is not a sos.



How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=[graph sos](#)

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BRST (2018): $\begin{array}{|c|c|} \hline \text{---} & \\ \hline \bullet & \bullet \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \end{array} \geq 0$ is a nonnegative graph polynomial that cannot be written as a graph sos.

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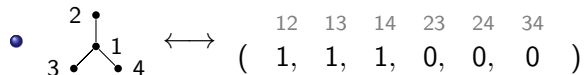
How? We characterize exactly which homogeneous graph polynomials of degree three can be written as a graph sos.

Tools to work on such problems

- Graphs on n vertices \longleftrightarrow subsets of $\{0, 1\}^{\binom{n}{2}}$

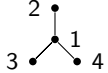
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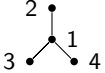
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-  \longleftrightarrow
$$\begin{pmatrix} & 12 & 13 & 14 & 23 & 24 & 34 \\ \begin{pmatrix} 1, & 1, & 1, & 0, & 0, & 0 \end{pmatrix} \end{pmatrix}$$

- Variables x_{ij}

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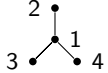
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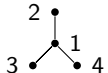
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- $x_{12} = \begin{matrix} 1 \\ 2 \end{matrix}$ and $x_{12}x_{13}x_{23} = \begin{matrix} & 1 \\ & \bullet \\ 2 & \bullet & 3 \end{matrix}$

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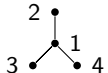
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- $x_{12}x_{13}x_{23}(G) = \begin{smallmatrix} & 1 \\ 2 & \text{---} & 3 \end{smallmatrix} (G)$ gives 1 if the vertices 1, 2, and 3 form a triangle in G , and 0 otherwise

Symmetrization

Example (Definition by example)

Let $\triangle = \text{sym}_n(\begin{smallmatrix} & 1 \\ 2 \triangle 3 \end{smallmatrix}) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(\begin{smallmatrix} & 1 \\ 2 \triangle 3 \end{smallmatrix})$.

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Example (Crucial definition by example: using only a subgroup of S_n)

Let ${}^1 \downarrow = \text{sym}_{\sigma \in S_n: \sigma \text{ fixes } 1}({}^1 \downarrow_2) = \frac{1}{n-1} \sum_{j \geq 2} x_{1j}$

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Example (One more example to clarify)

$$\begin{smallmatrix} & 1 \\ 2 \triangle 3 \end{smallmatrix} \left(\begin{smallmatrix} 3 & & 5 \\ & 1 & 2 \\ 4 & & 6 \end{smallmatrix} \right) = \frac{2}{4}$$

Miracle 1: (asymptotic) multiplication

$$\begin{aligned} \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} &= \frac{1}{(n-1)^2} \left(\sum_{j \geq 2} x_{1j} \right)^2 \\ &= \frac{1}{(n-1)^2} \sum_{j \geq 2} x_{1j}^2 + \frac{2}{(n-1)^2} \sum_{2 \leq i < j} x_{1i} x_{1j} \\ &= \frac{1}{(n-1)^2} \sum_{j \geq 2} x_{1j} + \frac{2}{(n-1)^2} \sum_{2 \leq i < j} x_{1i} x_{1j} \\ &\approx \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} \end{aligned}$$

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The diagram is a graph with three vertices. Two vertices at the bottom are connected by an edge. A third vertex is positioned above the midpoint of this edge and is connected to both bottom vertices by edges. The top vertex is labeled with the number '1'.

Multiplying asymptotically = gluing!

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Example

$$\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ 1 \bullet \quad \bullet 3 \\ \swarrow \quad \searrow \\ \bullet \quad \bullet 2 \end{array} \cdot \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ 2 \bullet \quad \bullet 1 \end{array} = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ 3 \bullet \quad \bullet 2 \\ \swarrow \quad \searrow \\ \bullet \quad \bullet 1 \end{array} \cdot \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet 2 \quad \bullet 1 \end{array} = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ 3 \bullet \quad \bullet 2 \\ \swarrow \quad \searrow \\ \bullet \quad \bullet 1 \end{array}$$

Certifying a nonnegative graph polynomial with a sos

Show that $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} - \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \geq 0$.

Certifying a nonnegative graph polynomial with a sos

Show that $\text{---}\diagup\diagdown\text{---} - \begin{smallmatrix} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{smallmatrix} \geq 0$.

$$\begin{aligned} \frac{1}{2} \text{sym}_n \left(\left(\begin{smallmatrix} 1 & \\ | & \end{smallmatrix} - \begin{smallmatrix} & 2 \\ & | \end{smallmatrix} \right)^2 \right) &= \frac{1}{2} \text{sym}_n \left(\begin{smallmatrix} 1 & 1 \\ | & | \end{smallmatrix} - 2 \begin{smallmatrix} 1 & 2 \\ | & | \end{smallmatrix} + \begin{smallmatrix} 2 & 2 \\ | & | \end{smallmatrix} \right) \\ &= \frac{1}{2} \text{sym}_n \left(\begin{smallmatrix} & 1 \\ \diagup & \diagdown \end{smallmatrix} - 2 \begin{smallmatrix} 1 & 2 \\ | & | \end{smallmatrix} + \begin{smallmatrix} & 2 \\ \diagup & \diagdown \end{smallmatrix} \right) \\ &= \frac{1}{2} (2 \begin{smallmatrix} & 1 \\ \diagup & \diagdown \end{smallmatrix} - 2 \begin{smallmatrix} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{smallmatrix}) \end{aligned}$$

Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d .

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Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d .

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$$\begin{aligned} & \text{sym}_n \left(\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} 1 \\ | \\ \bullet \end{array} \begin{array}{c} 4 \\ | \\ \bullet \end{array} - 1 \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} 2 \\ | \\ \bullet \end{array} \right)^2 + 2 \text{sym}_n \left(\begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 4 \\ | \\ 3 \end{array} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} 5 \\ | \\ \bullet \end{array} + (\sqrt{2}-1) \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} \begin{array}{c} 5 \\ | \\ 6 \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right)^2 \\ &= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - 2 \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \end{aligned}$$

Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d .

Example

$$\begin{aligned}
 & \text{sym}_n \left(\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 1 \\ | \\ \cdot \end{array} \begin{array}{c} 4 \\ | \\ \cdot \end{array} - 1 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} \right)^2 + 2 \text{sym}_n \left(\begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 4 \\ | \\ 3 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 5 \\ | \\ \cdot \end{array} + (\sqrt{2}-1) \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} \begin{array}{c} 5 \\ | \\ 6 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \right)^2 \\
 &= \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} - 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \\
 &= \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array}
 \end{aligned}$$

Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d .

Example

$$\begin{aligned}
 & \text{sym}_n \left(\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 1 \\ | \\ \cdot \end{array} \begin{array}{c} 4 \\ | \\ \cdot \end{array} - 1 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} \right)^2 + 2 \text{sym}_n \left(\begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 4 \\ | \\ 3 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 5 \\ | \\ \cdot \end{array} + (\sqrt{2}-1) \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 5 \\ | \\ 6 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \right)^2 \\
 &= \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} - 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \\
 &= \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} + 2 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} = \text{sym}_n \left(\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ 2 \quad 1 \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 1 \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} + 1 \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} \cdot \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} \begin{array}{c} 2 \\ | \\ \cdot \end{array} \right)^2
 \end{aligned}$$

All graph sums of squares of degree 3

Theorem (BRST 2018)

Any homogeneous graph sos of degree 3 can be written as

$$\begin{aligned} & \text{sym}_n \left(a_1 \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ 1 \bullet \quad \bullet \end{array} \begin{array}{c} 2 \\ \bullet \end{array} \right) + a_2 \begin{array}{c} 1 \\ \bullet \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right)^2 + \text{sym}_n \left(a_3 \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ 1 \bullet \quad \bullet \end{array} \begin{array}{c} 2 \\ \bullet \end{array} \right) \right)^2 \\ & + \text{sym}_n \left(a_4 \left(\begin{array}{c} 1 \\ \bullet \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 3 \\ \bullet \end{array} - \begin{array}{c} 1 \\ \bullet \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 4 \\ \bullet \end{array} \right) \right)^2 + \text{sym}_n \left(a_5 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} \begin{array}{c} 3 \\ \bullet \end{array} \right)^2 + \text{sym}_n \left(a_6 \begin{array}{c} 2 \\ \bullet \\ 3 \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} \begin{array}{c} 4 \\ \bullet \end{array} \right)^2 \\ & + \text{sym}_n \left(a_7 \begin{array}{c} 2 \\ \bullet \\ 1 \bullet \quad \bullet \end{array} \begin{array}{c} 3 \\ \bullet \end{array} \begin{array}{c} 4 \\ \bullet \end{array} \right)^2 + \text{sym}_n \left(a_8 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 1 \\ \bullet \end{array} \begin{array}{c} 3 \\ \bullet \end{array} \begin{array}{c} 4 \\ \bullet \\ 5 \bullet \end{array} \right)^2 + \text{sym}_n \left(a_9 \begin{array}{c} 1 \\ \bullet \\ 2 \bullet \quad \bullet \end{array} \begin{array}{c} 3 \\ \bullet \end{array} \begin{array}{c} 4 \\ \bullet \end{array} \begin{array}{c} 5 \\ \bullet \\ 6 \bullet \end{array} \right)^2 \text{ where} \\ & a_1, \dots, a_9 \in \mathbb{R}. \end{aligned}$$

All graph sums of squares of degree 3

Theorem (BRST 2018)

Any homogeneous graph sos of degree 3 can be written as

$$\begin{aligned} & \text{sym}_n \left(a_1 \left(\begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \end{array} + \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 1 \end{array} \right) + a_2 \begin{array}{c} 1 \quad 1 \\ \parallel \quad \parallel \\ 2 \end{array} \right)^2 + \text{sym}_n \left(a_3 \left(\begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \end{array} - \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 1 \end{array} \right) \right)^2 \\ & + \text{sym}_n \left(a_4 \left(\begin{array}{c} 1 \quad 3 \\ \parallel \quad \parallel \\ 2 \end{array} - \begin{array}{c} 1 \quad 4 \\ \parallel \quad \parallel \\ 2 \end{array} \right) \right)^2 + \text{sym}_n \left(a_5 \begin{array}{c} 1 \\ \diagup \quad \diagdown \quad \diagup \\ 2 \quad 3 \end{array} \right)^2 + \text{sym}_n \left(a_6 \begin{array}{c} 2 \\ \diagup \quad \diagdown \quad \diagup \\ 3 \quad 1 \quad 4 \end{array} \right)^2 \\ & + \text{sym}_n \left(a_7 \begin{array}{c} 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \right)^2 + \text{sym}_n \left(a_8 \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} \begin{array}{c} 4 \quad 4 \\ \parallel \quad \parallel \\ 5 \quad 5 \end{array} \right)^2 + \text{sym}_n \left(a_9 \begin{array}{c} 1 \quad 3 \quad 5 \\ \parallel \quad \parallel \quad \parallel \\ 2 \quad 4 \quad 6 \end{array} \right)^2 \text{ where} \\ & a_1, \dots, a_9 \in \mathbb{R}. \end{aligned}$$

Equivalently, it can be written as

$$\begin{aligned} & a \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \end{array} + (b + 4m_2 + f) \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \end{array} \begin{array}{c} 1 \\ \parallel \\ 2 \end{array} + (2m_1 + c + g) \begin{array}{c} 1 \\ \diagup \quad \diagdown \quad \diagup \\ 2 \quad 3 \end{array} \\ & + (2m_1 + d - g) \begin{array}{c} 1 \quad 1 \\ \parallel \quad \parallel \\ 2 \end{array} + (m_3 + e - f) \begin{array}{c} 1 \quad 1 \quad 1 \\ \parallel \quad \parallel \quad \parallel \\ 2 \end{array} \\ & \text{where } a, b, c, d, e, f, g \geq 0 \text{ and } \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \succeq 0. \end{aligned}$$

Corollary (BRST 2018)

$a \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet \\ \hline \end{array} \geq 0$ is not a sum of squares for any $a \in \mathbb{R}$.

Thank you!

Also follow `_forall` on instagram
or check out www.instagram.com/_forall.

3-profiles of graphs

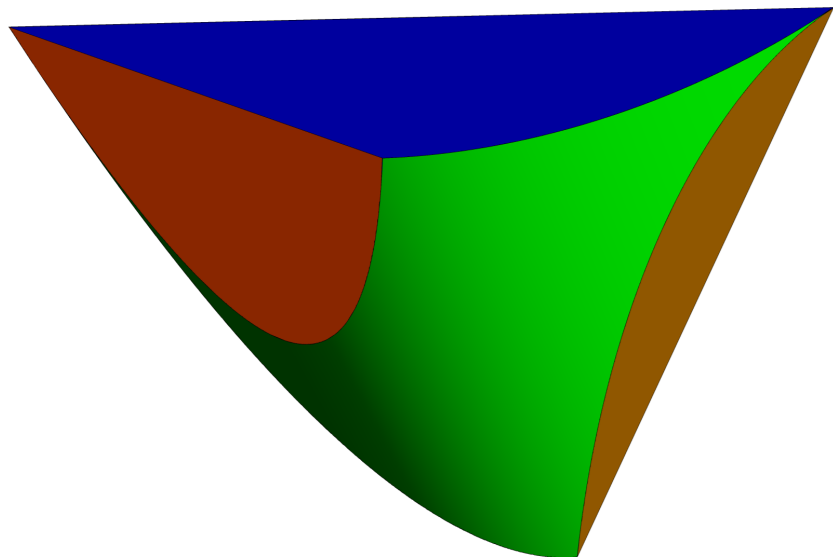
BRST(2018):

$(d(\bullet \overset{\bullet}{\cdot} \bullet, G), d(\bullet \text{---} \bullet, G), d(\bullet \text{---} \bullet \text{---} \bullet, G), d(\bullet \text{---} \bullet \text{---} \bullet, G))$ is contained in

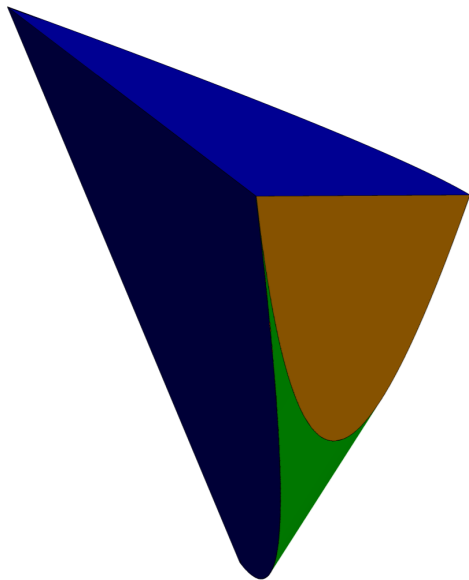
$$B = \{x \in \mathbb{R}^4 : x_0 + x_1 + x_2 + x_3 = 1, \\ x_0, x_1, x_2, x_3 \geq 0 \\ \begin{pmatrix} 3x_0 + x_1 & x_1 + x_2 \\ x_1 + x_2 & x_2 + 3x_3 \end{pmatrix} \succeq 0\}$$

which looks like...

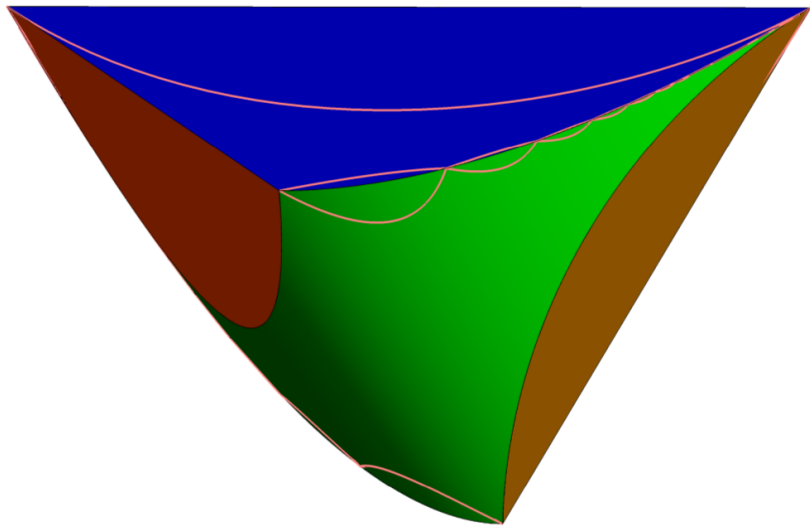
Convex relaxation for 3-profiles of graphs



Convex relaxation for 3-profiles of graphs



Actual 3-profiles of graphs



Actual 3-profiles of graphs

