Large graphs and symmetric sums of squares

Annie Raymond
joint with Greg Blekherman, Mohit Singh and Rekha Thomas

University of Massachusetts, Amherst

October 18, 2018

An example

Theorem (Mantel, 1907)

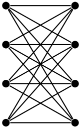
The maximum number of edges in a graph on n vertices with no triangles is $\lfloor \frac{n^2}{4} \rfloor$*. In particular, as* $n \to \infty$*, the maximum edge density goes to* $\frac{1}{2}$*.*

An example

Theorem (Mantel, 1907)

The maximum number of edges in a graph on n vertices with no triangles is $\lfloor \frac{n^2}{4} \rfloor$. In particular, as $n \to \infty$, the maximum edge density goes to $\frac{1}{2}$.

The maximum is attained on $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$:

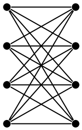


An example

Theorem (Mantel, 1907)

The maximum number of edges in a graph on n vertices with no triangles is $\lfloor \frac{n^2}{4} \rfloor$. In particular, as $n \to \infty$, the maximum edge density goes to $\frac{1}{2}$.

The maximum is attained on $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$:



$\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$ edges out of $\binom{n}{2}$ potential edges, no triangles.

Other triangle densities

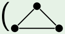
What if I want to know the maximum edge density in a graph on n vertices with a **triangle density of y**, for some $0 \leq y \leq 1$?

Other triangle densities

What if I want to know the maximum edge density in a graph on n vertices with a **triangle density of y**, for some $0 \leq y \leq 1$?

Example

Let $G =$ ,

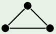
then $(d($$, G), d($$, G)) = \left(\frac{9}{\binom{7}{2}}, \frac{2}{\binom{7}{3}}\right) \approx (0.43, 0.06)$.

Other triangle densities

What if I want to know the maximum edge density in a graph on n vertices with a **triangle density of y**, for some $0 \leq y \leq 1$?

Example

Let $G =$ ,

then $(d($$, G), d($$, G)) = \left(\frac{9}{\binom{7}{2}}, \frac{2}{\binom{7}{3}}\right) \approx (0.43, 0.06)$.

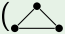
Is that the max edge density among graphs on 7 vertices with 2 triangles?

Other triangle densities

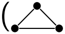
What if I want to know the maximum edge density in a graph on n vertices with a **triangle density of y**, for some $0 \leq y \leq 1$?

Example

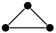
Let $G =$ ,

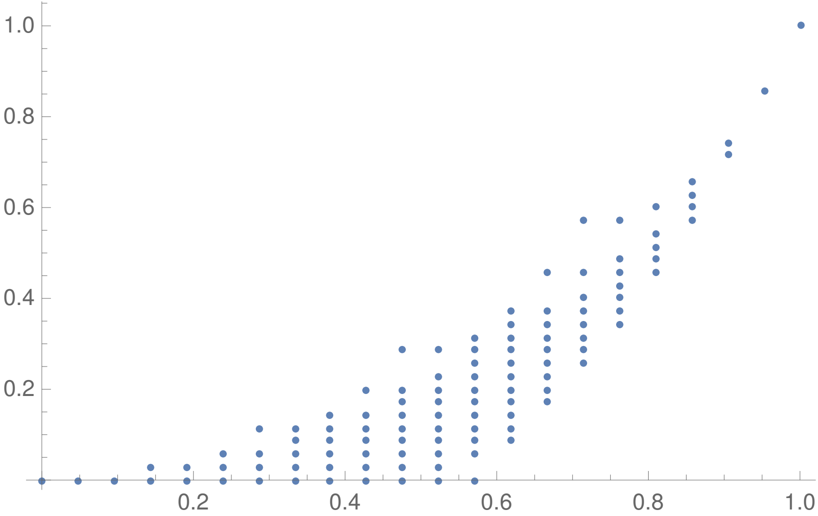
then $(d($$, G), d($$, G)) = \left(\frac{9}{\binom{7}{2}}, \frac{2}{\binom{7}{3}}\right) \approx (0.43, 0.06)$.

Is that the max edge density among graphs on 7 vertices with 2 triangles?

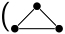
What can $(d($$, G), d($$, G))$ **be if** G **is any graph on** 7 **vertices?**

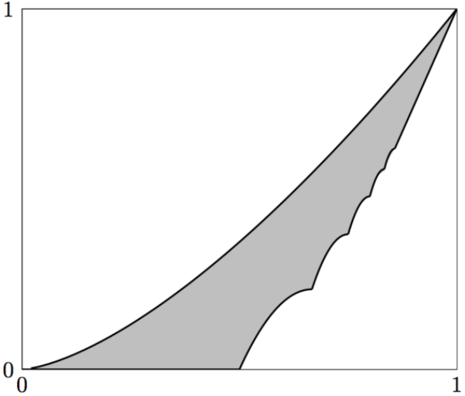
All density vectors for graphs on 7 vertices

$(d($$, G), d($$, G))$ for any graph G on 7 vertices



All density vectors for graphs on n vertices as $n \to \infty$

$(d($$, G), d($$, G))$ for any graph G on n vertices as $n \to \infty$



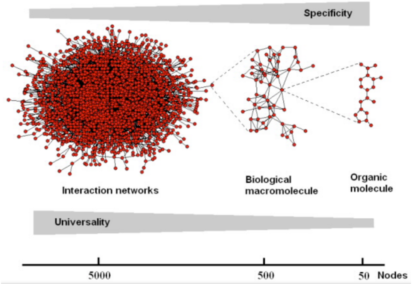
(Razborov, 2008)

Why care?

Large graphs are everywhere!

Why care?

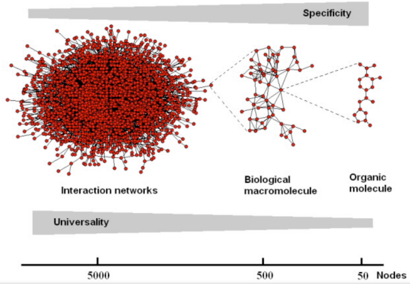
Large graphs are everywhere!



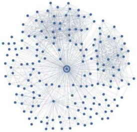
Biology

Why care?

Large graphs are everywhere!

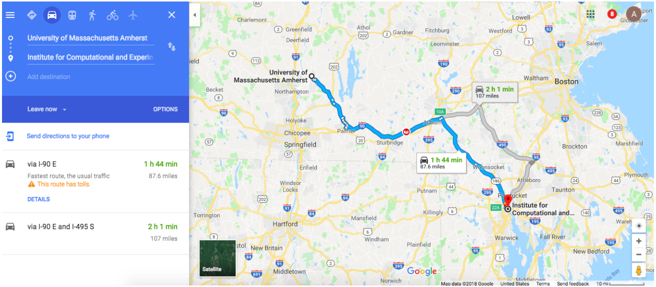


Biology



Facebook graph

More reasons to care!



University of Massachusetts Amherst

Institute for Computational and Experim

Add destination

Leave now

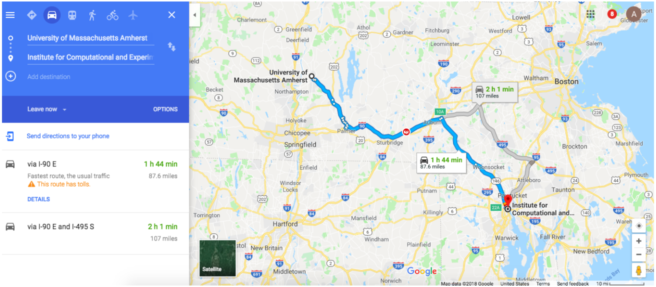
OPTIONS

Send directions to your phone

via I-90 E	1 h 44 min
Fastest route, the usual traffic	87.6 miles
⚠ This route has tolls.	
DETAILS	
via I-90 E and I-495 S	2 h 1 min
	107 miles

Google Maps

More reasons to care!



University of Massachusetts Amherst

Institute for Computational and Experin

Add destination

Leave now

OPTIONS

Send directions to your phone

via I-90 E — 1 h 44 min — 87.6 miles

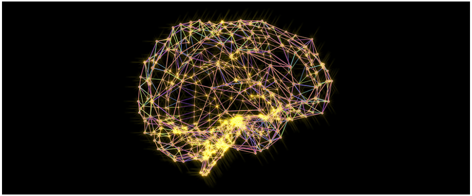
Fastest route, the usual traffic

This route has tolls.

DETAILS

via I-90 E and I-495 S — 2 h 1 min — 107 miles

Google Maps



Alfred Pasieka/Science Photo Library/Getty Images

Problem

Those graphs are sometimes too large for computers!

Problem

Those graphs are sometimes too large for computers!

Idea: understand the graph locally

Problem

Those graphs are sometimes too large for computers!

Idea: understand the graph locally

This raises immediately two questions:

Problem

Those graphs are sometimes too large for computers!

Idea: understand the graph locally

This raises immediately two questions:

1. How do global and local properties relate?

Problem

Those graphs are sometimes too large for computers!

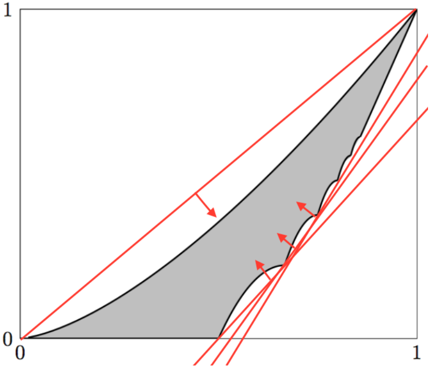
Idea: understand the graph locally

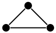
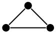
This raises immediately two questions:

1. How do global and local properties relate?
2. What is even possible locally?

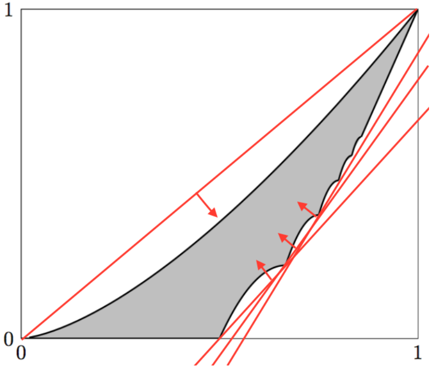
Graph density inequalities

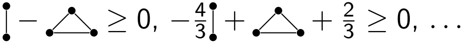
Graph density inequalities



 $-$  ≥ 0, $-\frac{4}{3}$  $+$  $+ \frac{2}{3} \geq 0, \ \ldots$

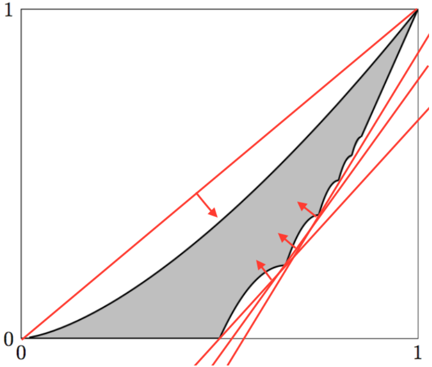
Graph density inequalities

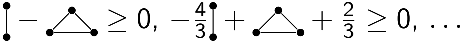




Nonnegative polynomial graph inequality: a polynomial* involving **any** graph densities (not just edges and triangles, and not necessarily just two of them) that, when evaluated on any graph on n vertices where $n \to \infty$, is nonnegative.

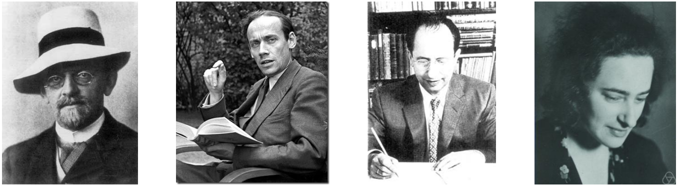
Graph density inequalities



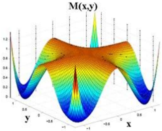


Nonnegative polynomial graph inequality: a polynomial* involving **any** graph densities (not just edges and triangles, and not necessarily just two of them) that, when evaluated on any graph on n vertices where $n \to \infty$, is nonnegative. **How can one certify such an inequality?**

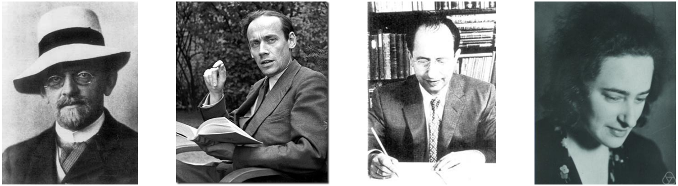
Certifying polynomial inequalities



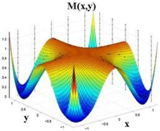
Certifying polynomial inequalities



A polynomial $p \in \mathbb{R}[x_1, \dots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \dots, x_n) \geq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

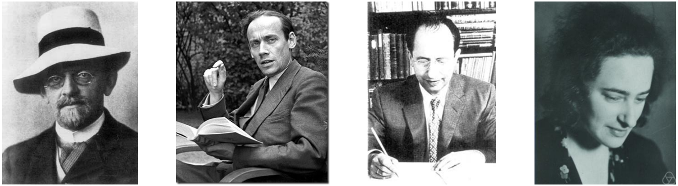


Certifying polynomial inequalities

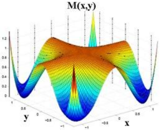


A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

p sum of squares (sos), i.e., $p = \sum_{i=1}^{l} f_i^2$ where $f_i \in \mathbb{R}[\mathbf{x}]$ $\Rightarrow$ $p \geq 0$



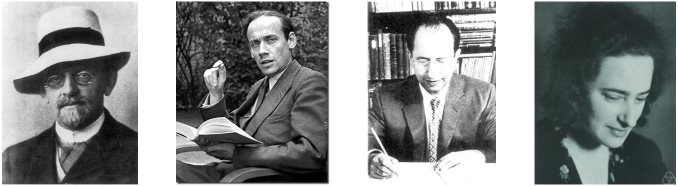
Certifying polynomial inequalities



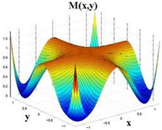
A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

p sum of squares (sos), i.e., $p = \sum_{i=1}^{l} f_i^2$ where $f_i \in \mathbb{R}[\mathbf{x}] \;\Rightarrow\; p \geq 0$

Hilbert (1888): Not all nonnegative polynomials are sos.



Certifying polynomial inequalities

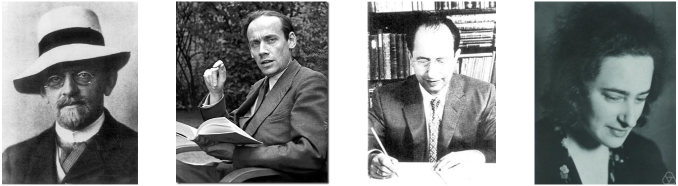


A polynomial $p \in \mathbb{R}[x_1, \dots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \dots, x_n) \geq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

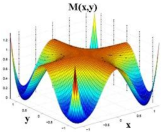
p sum of squares (sos), i.e., $p = \sum_{i=1}^{l} f_i^2$ where $f_i \in \mathbb{R}[\mathbf{x}] \;\Rightarrow\; p \geq 0$

Hilbert (1888): Not all nonnegative polynomials are sos.

Artin (1927): Every nonnegative polynomial can be written as a sum of squares of rational functions.



Certifying polynomial inequalities



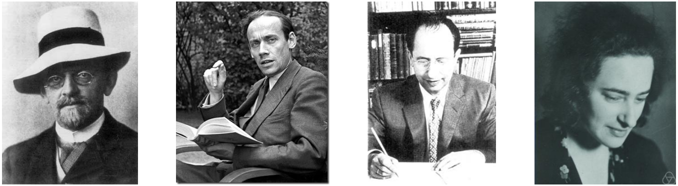
A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

p sum of squares (sos), i.e., $p = \sum_{i=1}^{l} f_i^2$ where $f_i \in \mathbb{R}[\mathbf{x}] \;\Rightarrow\; p \geq 0$

Hilbert (1888): Not all nonnegative polynomials are sos.

Artin (1927): Every nonnegative polynomial can be written as a sum of squares of rational functions.

Motzkin (1967, with Taussky-Todd): $M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is a nonnegative polynomial but is not a sos.



How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

Hatami-Norine (2011): Not every nonnegative graph polynomial can be written as a graph sos

How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

Hatami-Norine (2011): Not every nonnegative graph polynomial can be written as a graph sos or even as a rational graph sos.

How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

Hatami-Norine (2011): Not every nonnegative graph polynomial can be written as a graph sos or even as a rational graph sos.

Lovász-Szegedy (2006) + Netzer-Thom (2015): Every nonnegative graph polynomial plus any $\epsilon > 0$ can be written as a graph sos.

How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

Hatami-Norine (2011): Not every nonnegative graph polynomial can be written as a graph sos or even as a rational graph sos.

Lovász-Szegedy (2006) + Netzer-Thom (2015): Every nonnegative graph polynomial plus any $\epsilon > 0$ can be written as a graph sos.

BRST (2018):  $-$  ≥ 0 is a nonnegative graph polynomial that cannot be written as a graph sos.

How do sums of squares fare with graph densities?

Sums of squares of polynomials involving graph densities=graph sos

Hatami-Norine (2011): Not every nonnegative graph polynomial can be written as a graph sos or even as a rational graph sos.

Lovász-Szegedy (2006) + Netzer-Thom (2015): Every nonnegative graph polynomial plus any $\epsilon > 0$ can be written as a graph sos.

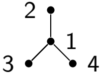
BRST (2018):  $-$  ≥ 0 is a nonnegative graph polynomial that cannot be written as a graph sos.

How? We characterize exactly which homogeneous graph polynomials of degree three can be written as a graph sos.

Tools to work on such problems

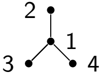
- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$

Tools to work on such problems

- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
- 

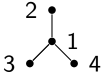
$$\longleftrightarrow \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ (\;1, & 1, & 1, & 0, & 0, & 0\;) \end{array}$$

Tools to work on such problems

- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
- 

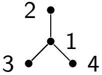
$\longleftrightarrow$ $\left(\begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ 1, & 1, & 1, & 0, & 0, & 0 \end{array}\right)$
- Variables x_{ij}

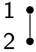
Tools to work on such problems

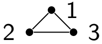
- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
- 

$\longleftrightarrow$ $\begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \left(\begin{array}{cccccc} 1, & 1, & 1, & 0, & 0, & 0 \end{array} \right) \end{array}$
- Variables x_{ij} $\rightarrow$ transform polynomials into pictures!

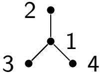
Tools to work on such problems

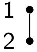
- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
- 

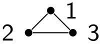
$\longleftrightarrow$ $\begin{array}{cccccccc} & 12 & 13 & 14 & 23 & 24 & 34 & \\ (& 1, & 1, & 1, & 0, & 0, & 0 &) \end{array}$
- Variables x_{ij} → transform polynomials into pictures!
- $x_{12} =$ 

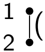
and $x_{12}x_{13}x_{23} =$ 

Tools to work on such problems

- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
- 

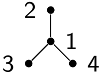
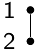
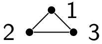
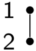
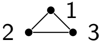
$\longleftrightarrow$ $\begin{array}{cccccccc} & 12 & 13 & 14 & 23 & 24 & 34 & \\ (& 1, & 1, & 1, & 0, & 0, & 0 &) \end{array}$
- Variables x_{ij} → transform polynomials into pictures!
- $x_{12} =$ 

and $x_{12}x_{13}x_{23} =$ 

- $x_{12}(G) =$ 

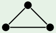
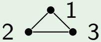
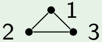
(G) gives 1 if $\{1,2\} \in E(G)$, and 0 otherwise

Tools to work on such problems

- Graphs on n vertices $\longleftrightarrow$ subsets of $\{0,1\}^{\binom{n}{2}}$
-  $\longleftrightarrow$ $\begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ (\,1, & 1, & 1, & 0, & 0, & 0\,) \end{array}$
- Variables x_{ij} → transform polynomials into pictures!
- $x_{12} =$  and $x_{12}x_{13}x_{23} =$ 
- $x_{12}(G) =$  (G) gives 1 if $\{1,2\} \in E(G)$, and 0 otherwise
- $x_{12}x_{13}x_{23}(G) =$  (G) gives 1 if the vertices 1,2, and 3 form a triangle in G, and 0 otherwise

Symmetrization

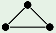
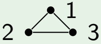
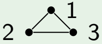
Example (Definition by example)

Let  $= \mathrm{sym}_n($  $) = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma($  $)$.

 (G) returns the triangle density of G.

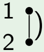
Symmetrization

Example (Definition by example)

Let  $= \mathrm{sym}_n($  $) = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma($  $)$.

 (G) returns the triangle density of G.

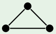
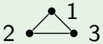
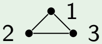
Example (Crucial definition by example: using only a subgroup of S_n)

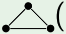
Let  $= \mathrm{sym}_{\sigma\in S_n:\sigma \text{ fixes } 1}($  $) = \frac{1}{n-1}\sum_{j\geq 2} x_{1j}$

 (G) returns the relative degree of vertex 1 in G.

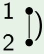
Symmetrization

Example (Definition by example)

Let  $= \text{sym}_n($  $) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma($  $)$.

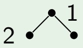
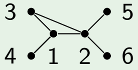
(G) returns the triangle density of G.

Example (Crucial definition by example: using only a subgroup of S_n)

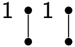
Let  $= \text{sym}_{\sigma \in S_n : \sigma \text{ fixes } 1}($  $) = \frac{1}{n-1} \sum_{j \geq 2} x_{1j}$

 (G) returns the relative degree of vertex 1 in G.

Example (One more example to clarify)

 $\Bigg($  $\Bigg) = \frac{2}{4}$

Miracle 1: (asymptotic) multiplication

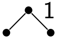


$$= \frac{1}{(n-1)^2}\left(\sum_{j\geq 2} x_{1j}\right)^2$$

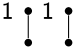
$$= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j}^2 + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j}$$

$$= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j} + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j}$$

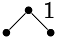
$$\approx$$



Miracle 1: (asymptotic) multiplication

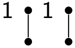


$$
\begin{aligned}
&= \frac{1}{(n-1)^2}\left(\sum_{j\geq 2} x_{1j}\right)^2 \\
&= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j}^2 + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j} \\
&= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j} + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j} \\
&\approx
\end{aligned}
$$



Multiplying asymptotically = gluing!

Miracle 1: (asymptotic) multiplication

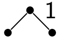


$$= \frac{1}{(n-1)^2}\left(\sum_{j\geq 2} x_{1j}\right)^2$$

$$= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j}^2 + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j}$$

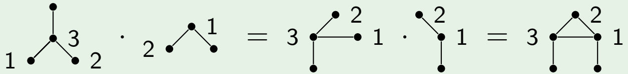
$$= \frac{1}{(n-1)^2}\sum_{j\geq 2} x_{1j} + \frac{2}{(n-1)^2}\sum_{2\leq i<j} x_{1i}x_{1j}$$

$$\approx$$



Multiplying asymptotically = gluing!

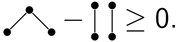
Example

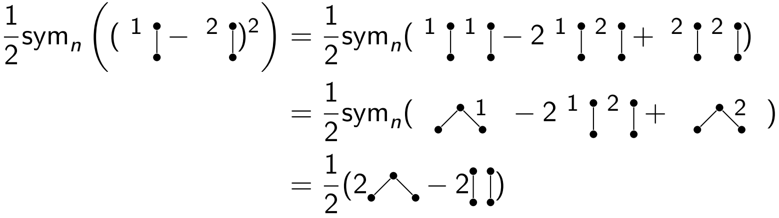


Certifying a nonnegative graph polynomial with a sos

Show that  $-$  ≥ 0.

Certifying a nonnegative graph polynomial with a sos

Show that 



Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d.

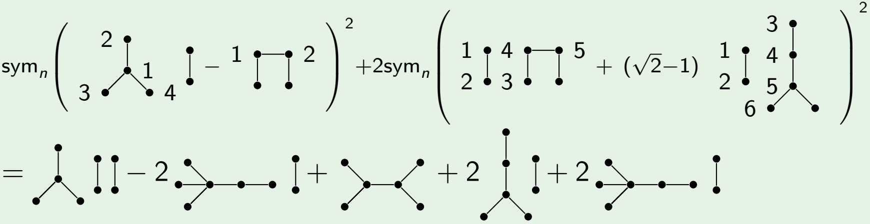
Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d.

Example



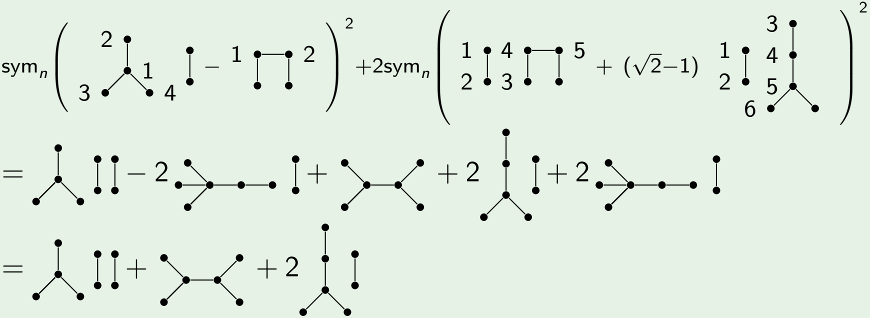
Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d.

Example



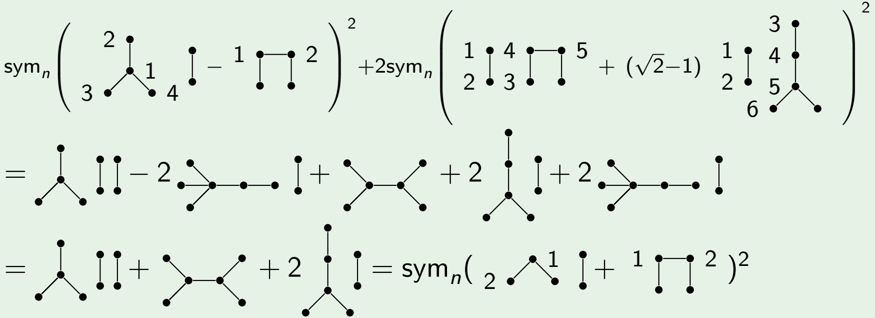
Miracle 2: homogeneous hegemony

Theorem (BRST 2018)

Consider a homogeneous nonnegative graph polynomial p of degree d that can be written as a graph sos.

Then p can be written out as a graph sos where any two monomials in any given square multiply to have degree d.

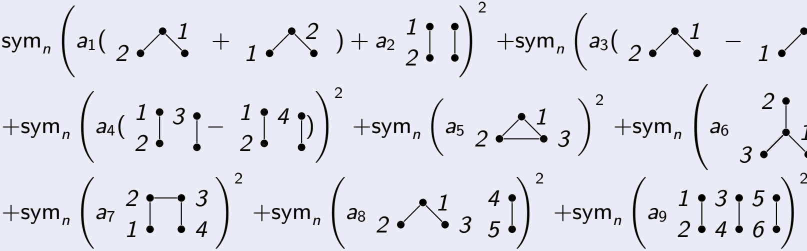
Example



All graph sums of squares of degree 3

Theorem (BRST 2018)

Any homogeneous graph sos of degree 3 can be written as

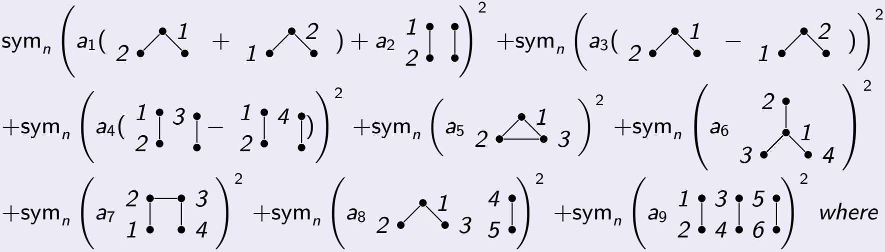


where $a_1, \dots, a_9 \in \mathbb{R}$.

All graph sums of squares of degree 3

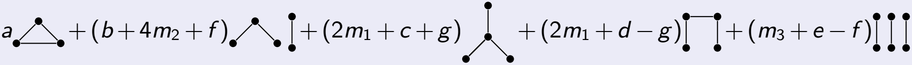
Theorem (BRST 2018)

Any homogeneous graph sos of degree 3 can be written as



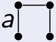
where $a_1, \ldots, a_9 \in \mathbb{R}$.

Equivalently, it can be written as



where $a, b, c, d, e, f, g \geq 0$ *and* $\begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \succeq 0$.

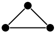
Corollary (BRST 2018)

a  $-$  ≥ 0 *is not a sum of squares for any* $a \in \mathbb{R}$.

Thank you!

Also follow _forall on instagram
or check out `www.instagram.com/_forall`.

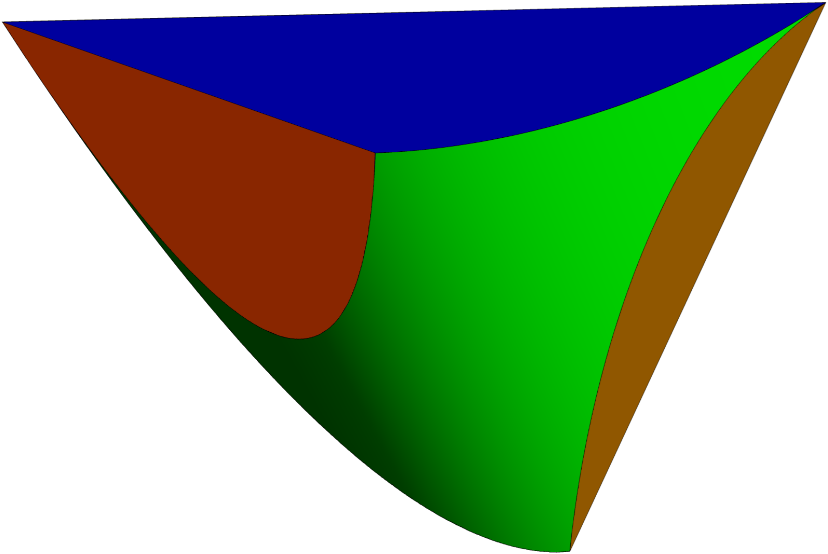
3-profiles of graphs

BRST(2018):
$(d($$, G), d($$, G), d($$, G), d($$, G))$ is contained in

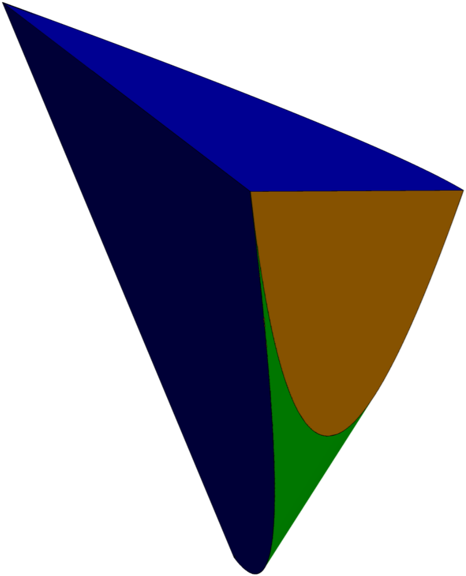
$$B = \{x \in \mathbb{R}^4 \colon x_0 + x_1 + x_2 + x_3 = 1,$$
$$x_0, x_1, x_2, x_3 \geq 0$$
$$\begin{pmatrix} 3x_0 + x_1 & x_1 + x_2 \\ x_1 + x_2 & x_2 + 3x_3 \end{pmatrix} \succeq 0\}$$

which looks like...

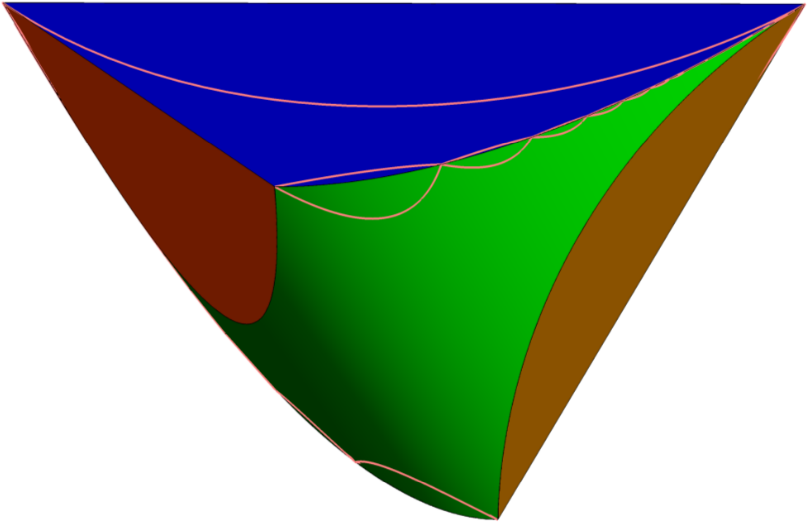
Convex relaxation for 3-profiles of graphs



Convex relaxation for 3-profiles of graphs



Actual 3-profiles of graphs



Actual 3-profiles of graphs